Considerations for Teaching with Multiple Methods: A Case Study of Missing-value Problems in Proportionality

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In this paper, we present a case study of a secondary mathematics teacher, Isaac (pseudonym), and his considerations for teaching with multiple methods for solving missing-value problems. While his students preferred methods that drew more closely on their intuitive understanding of proportionality, Isaac emphasised the algorithmic cross-multiplication method. Analysis of Isaac's introduction and use of the cross-multiplication method suggest his key considerations were linked demonstrating the efficiency of the cross-multiplication method, while also helping students to make making meaning from the cross-multiplication method.

Teaching with multiple methods is widely recognised as beneficial for developing students' conceptual understanding and procedural fluency (Durkin et al., 2017). In the context of proportionality, a topic which is fundamental in mathematics, demonstrating multiple methods helps to develop students' proportional reasoning (Cramer & Post, 1993). This requires teachers to facilitate instruction that draws on the connections across methods and to explicitly discuss the multiplicative relationships between quantities (Fernandez et al., 2010). While some methods are more intuitive for students to understand, such as using the scalar factor between two quantities to solve for a missing value, the use of the cross-multiplication method is still prevalent despite its lack of meaning to students. In this study, we examine the case of a teacher who taught with three methods for solving missing-value problems: using the multiplicative factor between ratios, using the multiplicative factor within ratios, and the cross-multiplication method, of which he spent significant lesson time explaining and demonstrating the cross-multiplication method. Hence, the research question that guided this study was: *What considerations did this teacher have for teaching with multiple methods in proportional problems?*

Background Literature

Teaching with Multiple Methods

Research has reported that teachers perceive several advantages in teaching with multiple methods. In a study conducted by Lynch and Star (2014) with middle- and high-school mathematics teachers (n = 92), the most cited advantage was that demonstrating multiple methods could cater to students' individual differences. As students have different learning styles and come with different background knowledge, these teachers wanted to ensure their students could see methods that were understandable to them and that resembled their own thinking during lessons. Another consideration teachers had was that they felt multiple methods helped to deepen students' understanding through developing their problem solving and reasoning skills. However, as Durkin et al. (2017) noted, demonstrating multiple methods is unproductive unless teachers compare and discuss them with students. This includes not only making sense of alternative methods and their similarities and differences, but also helping students to recognise the affordances of certain strategies over others relative to the context

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and making connections with other concepts. They concluded that doing so successfully requires careful selection of strategies and problems for comparison.

One way this can be achieved is through the design of instructional materials, a common practice for Singapore mathematics teachers. Through the design of instructional materials, such as worksheets, teachers can embed their instructional goals into the selection, modification, and design of items (e.g., worked examples, helpful hints, practice questions) that will be implemented in their lessons. This was achieved by an experienced and competent teacher in a case study reported by Toh et al. (2021), who wanted to help students make connections across different representations and strategies for quadratic equations. She implemented tasks that presented multiple methods for solving a single quadratic equation and required students to choose a method to adopt and to explain why. This was followed by another activity where she provided three quadratic equations and called on students to demonstrate their preferred method for each one along with justification of their choice of method. As noted by Cramer and Post (1993), helping students to see and explain multiple methods is also encouraged in the teaching of proportionality to develop students' proportional reasoning and ability to see multiplicative relationships.

Teaching Proportionality

Proportionality is a fundamental topic in mathematics that is highly connected to several concepts (Lamon, 2007). Notably, the recent emphasis on *Big Ideas* in the 2020 secondary mathematics syllabus in Singapore listed eight clusters of ideas, of which proportionality is one. In his review of the literature on research and the teaching of proportionality, Yeo (2019) defined proportionality to mean the existence of a constant ratio between two quantities, which includes both direct and indirect proportionality. He clarified that two key ideas in proportionality were the equality of two ratios ($\frac{y_2}{y_1} = \frac{x_2}{x_1}$ for direct proportion) and the equality of two rates ($\frac{y_2}{x_2} = \frac{y_1}{x_1}$ for direct proportion). In the 2020 secondary mathematics syllabus, this is described more functionally as the "relationship between two quantities that allows one quantity to be computed from the other based on multiplicative reasoning" (Ministry of Education, 2019, p. 8).

Within the topic of proportionality, Lamon (2007) determined two main types of problems: missing-value problems (e.g., $1{:}\,4 \text{ is equivalent to } 5{:}\,x. \text{ Find } x$) and comparison problems (e.g., Drink A is mixed with 3 parts juice and 8 parts water, Drink B is mixed with 4 parts juice and 9 parts water. Which drink is sweeter?). For missing-value problems, several researchers have documented the different methods that students and teachers adopt. Artut and Pelen (2015) reported that using the *factor of change* (i.e., determining the multiplicative factor represented by the equality of ratios and rates) was the most common method adopted by sixth-grader students. In comparison, Cramer and Post (1993) note that the *cross-multiplication algorithm* is the most commonly taught method in textbooks, but that the *unitary-rate method* was the most commonly used amongst seventh-grade students, followed by the factor of change which were both perceived by students as being more intuitive. In their study of the strategies that preservice middle- and high-school teachers used to solve proportion problems, Arican (2018) found that the preservice teachers actively avoided using the cross-multiplication method, when it came to more complex problems with multiple proportions and when the multiplicative relationships were not clear.

Across all of these studies, a common issue discussed was that while the cross-multiplication method was efficient, it has no meaning for students (Cramer & Post, 1993), and hence students and teachers have been found to apply it incorrectly or inappropriately with non-proportional contexts. The common consensus is that the cross-multiplication method is often taught in such a way that it doesn't require students to reflect on the number structure

(i.e., the proportionality). Hence, it is suggested that while using multiple methods to teach proportionality is recommended, methods such as the unitary-rate and factors of change should take precedence to ensure students engage in proportional reasoning.

In this study, we explore the case of a secondary mathematics teacher, Isaac (pseudonym), who taught missing-value problems with multiple strategies but appeared to emphasise the importance of the cross-multiplication method. Given the existing literature on the disadvantages of teaching the cross-multiplication method, as students would likely apply it as an algorithm with no meaning or overuse it in inappropriate situations, this led us to wonder, why was the cross-multiplication method important to teach along with other methods? Despite alternative methods that have been shown to be more intuitive for students for engaging in proportional reasoning, what affordances did Isaac perceive with teaching the cross-multiplication algorithm that warranted its special feature across the two lessons?

Methods

The data presented in this paper is part of a larger study about mathematics teachers' design of instructional materials. Four teachers from two secondary schools in Singapore engaged in professional learning community (PLC) discussions at their respective schools with their colleagues, individual design of a set of worksheets on the topics of Ratio and Rates, and one-on-one semi-structured interviews with the first author about their worksheet design (e.g., selection, modification, creation of tasks). Then, they implemented their worksheets with their students and engaged in post-lesson one-on-one semi-structured interviews to reflect on their lessons. All worksheet drafts and teacher notes were collected, and all PLC discussions, interviews, and lessons were video-recorded and selected episodes were transcribed.

In this paper, we discuss the data of one teacher, Isaac (pseudonym), and the implementation of his first worksheet on ratios over two lessons. Of the four teachers, Isaac frequently demonstrated the use of multiple methods in his lessons, with special attention given to the cross-multiplication method, while the other teachers tended to adopt more advantageous and canonical methods. Our aim was to determine Isaac's considerations for teaching with multiple methods, which we analysed from his implementation on two levels of grain-size. At the item-level, we evaluated the mathematical content (e.g., relevant concepts, processes, possible solution methods) in each item in Isaac's worksheets and their potential affordances. At the set-level, we examined the connections of the content and affordances between items, and Isaac's overarching instructional goals. Adopting these two grain-sizes highlighted the unique role of individual tasks in developing students' understanding of proportionality, while also capturing how Isaac used these tasks collectively to deepen students' understanding of proportionality.

Results and Discussion

In this section, we begin by presenting a description of two tasks from Isaac's implementation and the questions that emerged from our item-level analysis about his considerations for teaching with multiple methods. As the cross-multiplication method emerged as a key method, we analyse a particular task that Isaac spent a significant portion of time discussing in the lesson at the item-level and set-level that will shed light on why Isaac emphasised this method.

Establishing Three Methods for Identifying Equivalent Ratios

While most of the items in Isaac's Ratio worksheet resembled "typical problems" found in textbooks, one item near the beginning of the worksheet caught our attention—a table of "Three methods for identifying equivalent ratios" (Figure 1), which Isaac had designed himself. This

table was intended to be a summary of methods that students could adopt to solve the problems in his worksheet and to discuss the different multiplicative relationships when dealing with equivalent ratios. As students were likely to have encountered Method 1 (M1, multiply by both side) and Method 2 (M2, multiplicative nature) in primary school, they responded confidently when Isaac asked them to determine the missing values in the examples using the multiplicative relationship between (M1) and within (M2) the equivalent ratios. These two methods use the *factor of change* (Lamon, 2007) to determine the missing value, which has been found to be a common strategy used by primary-level and early secondary students (Artut & Pelen, 2015). As students could clearly articulate that the missing values would be "five times" and "four times" another quantity, Isaac spent little time elaborating on these and swiftly moved on to introduce the third method.

When Isaac introduced Method 3 (M3, cross-multiply), he had difficulty communicating to students the meaning behind the procedure. Prior to the lesson, he anticipated that students would not immediately understand why cross-multiplying the ratios could be used to identify missing values. Isaac asked them if they realised that the products would be equal when they cross-multiplied the quantities, $1 \times 8 = 2 \times 4$. When he asked the class to think about why, the students said it was because "the quantities are multiplied by 2 on both sides" and "the second quantity is a quarter of the first". These were not the reasons that Isaac hoped students would use. He attempted to resolve this by demonstrating M3 on the same missing-value problem in M2, to show that when they cross-multiplied the values for the ratios $1{:}\,4$ and $x{:}\,A$, they would arrive at the same solution as M2. Isaac's demonstration appeared to have little effect on the students; they remained silent and appeared to be unsure of why it was equal still. After the lesson he recounted that in the moment, "it's even difficult for me as the teacher to answer as well, I have to admit that". Isaac acknowledged that he only demonstrated the procedure and hadn't provided a proper rationale; hence, he was not surprised that students were unconvinced with M3. For the next lesson, he planned to revisit all three methods, to demonstrate their use on practice questions, and to explain M3 again.

THREE METHODS TO IDENTIFY EQUIVALENT RATIOS

Methods	Example	Come up with your own example!
Multiply by both side	×5 1 : 4 2×5 5 : …20…	
Multiplicative nature	1 : 4 ×4 x : …4x… ×4	
'Cross-multiply' method	1: 4 2: 8 1 : 4 x : …4x. 1×A = 4×x A = 4x.	

Figure 1. "Three Methods to Identify Equivalent Ratios" from Isaac's lesson.

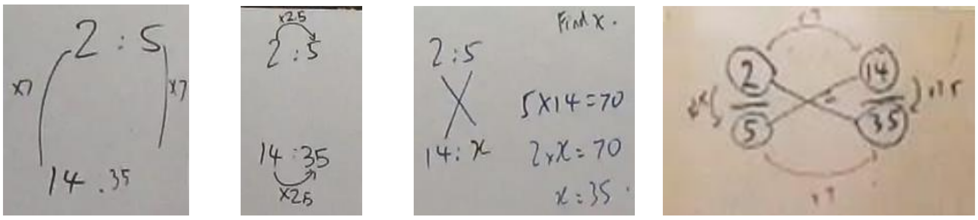
Isaac's implementation of the table, especially his teaching of M3, raised several questions about his considerations for teaching with multiple methods. Why did he want to introduce M3 when he was having difficulty communicating the underlying reasoning and suspected that students would not be able to intuitively make sense of it? Notably, his students' explanations

for M3 drew on the factor of change methods, M1 and M2, which Cramer and Post (1993) claimed are more naturally intuitive and meaningful for students than the cross-multiply method (M3). Although M1, M2, and the unitary method that is taught in primary school (Yeo, 2019) suffice to answer missing-value problems, what affordances did Isaac observe in M3? Further analysis of his implementation will shed light on his considerations.

Practicing Three Methods and Revisiting Method 3

After introducing the three methods, Isaac encouraged students to see how they could be applied across several problems in his worksheet. For all the questions in his Ratio worksheet, he invited students to write their solutions on the whiteboard or he would write the common methods that he observed amongst the class and asked the students to explain. When only one method was predominantly used by students—usually M1—Isaac typically called on students to describe additional methods and wrote these on the board. He frequently connected the three methods by pointing out that they produced the same solution due to the "proportionality" and "multiplicative relations" between the quantities in the ratios. However, he had not yet addressed the rationale for M3 or provided a reason for why students should try to understand and adopt it.

When Isaac attempted to address the rationale for M3 in the following lesson, he again encountered issues communicating the reasoning to students. At the beginning of the second lesson, Isaac generated a new missing-value problem, $2:5$ and $14:x$, and asked for three students to come to the board to demonstrate the three methods to find x. After each student came to the board to write their solutions (*Figure 2*, A-C), Isaac connected the three methods by explaining that "in all three methods, we claim that this ratio $(2:5)$ is *equivalent* to this ratio $(14:35)$ because the first quantity is *proportional* to the second quantity. In other words, I have *assumed* that the ratios are equivalent". Although the class could observe how to apply the three methods, only three students said M3 "made sense". Isaac explained that as the ratios could be expressed as equivalent fractions (*Figure 2*, D), by cross-multiplying not only would the products be the same but the multiplicative relationships between the quantities for M1 and M2 could be seen also. Isaac's students were struggling to make sense of M3, and even Isaac was having difficulty convincing students about the rationale.



(A) (B) (C) (D)

Figure 2. Isaac's task for revising the Three Methods in Lesson 2 (A, B, C by students, D by Isaac).

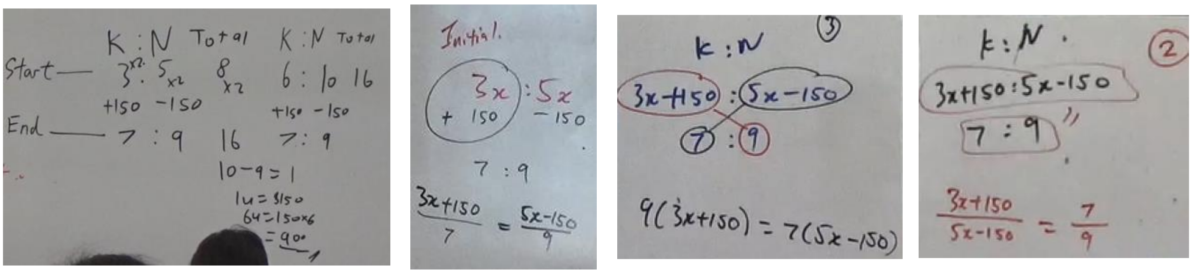
Isaac's continued attempts to demonstrate and explain M3 suggest that he thought it was an important method for students to understand—but *why*? While teaching with multiple strategies and helping students to see the relationships between quantities is beneficial for developing conceptual understanding about proportionality (Cramer & Post, 1993; Fernandez et al., 2010), Isaac's students can already use and explain at least three methods (M1, M2, unitary method) to solve ratio problems. It remains unclear what additional affordances Isaac saw in M3. What were Isaac's considerations for revisiting and *emphasising* M3, when alternative methods were sufficient, more intuitive, and more popular amongst his students?

As Isaac progressed through the problems he selected for his worksheet, the value of M3 became more apparent in one task near the end.

The Importance of Method 3: Cross-multiplication Method

In comparison to the other problems that were discussed in his lessons, Isaac's implementation of this problem about the exchange of money between two friends (Figure 3) began to shed light on his considerations for emphasising M3. When it initially came up near the end of Lesson 1, it was correctly solved by most of the students in the class using a unitary-rate approach (*Figure 3*, A) that reflected the corresponding worked-examples provided in the textbook from which Isaac sourced this problem. Although students had found the correct solution, Isaac's teacher notes showed that he wanted to discuss solutions that more closely resembled the three methods and involved expressing the ratios as fractions. One student in the class solved the problem by forming a fraction (*Figure 3*, B), so Isaac wrote it on the whiteboard. Noticing how his students were puzzled by the emergence of the equation $\frac{3x+150}{7} = \frac{5x-150}{9}$, when the end-of-lesson bell rang shortly after, he decided that he would revisit the problem the next day. Isaac acknowledged that using the unitary-rate method was easy for students to understand, but it was "super specialised"—suggesting there were some contexts where it might be inappropriate or inconvenient. In comparison, the other method (*Figure 2*, B) "is so incredibly difficult, at least for students at their level right now", but he wanted them to make sense of it as it would "help them to understand that ratios can be expressed as fractions, as equations, which in the future as they go to upper secondary in mensuration, they will need this."

Kate and Nora each have a sum of money. The ratio of the amount of money Kate has to that of Nora is $3:5$. *After Nora gives \$150 to Kate, the ratio of the amount of money Kate has to that of Nora becomes* $7:9$. *Find the sum of money Kate had initially.*



(A) (B) (C) (D)

Figure 3. Four methods for solving a problem (A and B by students; C and D by Isaac).

On his second attempt to explain the problem the next day, Isaac was successful in communicating to students the efficiency of M3 by making connections with alternative methods. For solution B, he added that as the ratio $3x + 150 : 5x - 150$ was equivalent to $7:9$, an equation could be formed to represent "1 part" by dividing $3x + 150$ and $5x - 150$ by 7 and 9 respectively. Then to solve for x, it naturally followed to cross-multiply to form the equation $9(3x + 150) = 7(5x - 150)$. Finally, the students accepted this explanation and Isaac moved on to demonstrate the direct use of M3 (*Figure 3*, C), showing that it would produce the same equation. Immediately after, Isaac explained how the two ratios could be directly expressed as equal fractions (*Figure 3*, D) and again mentioned how the next step would be to cross-multiply, resembling the same equation in M3. The students accepted that across these three solutions, they all required cross-multiplication and eventually all arrived at

the same algebraic equation $9(3x + 150) = 7(5x - 150)$ to solve for the total sum of money. In this task, Isaac had convinced students that M3 would be more efficient, while simultaneously making connections across other methods.

Our analysis of Isaac's implementation of this task surfaced two key considerations that Isaac considered for teaching with multiple methods and emphasising M3. At the item-level, this task demonstrated the *efficiency and generalisability* of M3. As the factors of change between ($\frac{7}{3x+150}$ or $\frac{9}{5x-150}$) and within ($\frac{5x-150}{3x+150}$ or $\frac{9}{7}$) the ratios are not immediately intuitive, applying M3 directly forms an algebraic equation which can be used to solve the problem. Instead of simply telling students to apply M3, Isaac made connections across the methods by discussing the general need for cross-multiplication, which would help students to see M3 as more than just a shortcut and to develop a more meaningful understanding of M3. However, research has reported on how the cross-multiplication method is often taught in textbooks and used by teachers as merely an algorithm without meaning (Arican, 2018; Cramer & Post, 1993). As a result, students typically avoid using it or use it inappropriately in non-proportional contexts due to a lack of understanding (Fernandez et al., 2010). Through forming different fractions and demonstrating how the equation $9(3x + 150) = 7(5x - 150)$ can be derived, Isaac imparted both efficiency and meaning to the method, and hence lessened the likelihood that students would simply apply M3 without being able to connect it to other methods.

In our set-level analysis, we determined that Isaac wanted to *build students' understanding* of M3 from their proficiency with M1 and M2. For the previous problems students encountered, students had no need to understand or use M3 because they had at least three methods they could use (M1, M2, unitary method). However, once students came to this problem, M3 became more than just another method for students to see; it became a key method for students to adopt both efficiently and meaningfully by building on their existing understanding of M1 and M2. As Isaac taught with multiple methods across his Ratio worksheet, including frequently inviting students to the board to write their solutions that included M1 and M2, his revisiting and re-explaining of M3 suggests he was deliberately preparing students to see the connections from one method to another. Although solving proportional problems using the cross-multiply method does not necessarily imply proportional reasoning has occurred (Arican, 2018; Lamon, 2007), as a collection of tasks where the Kate and Nora problem plays a unique role, the cross-multiply method became relevant to proportional reasoning with other tasks and through other methods.

Much like the teachers in Lynch and Star's (2014) study of teachers' considerations for teaching with multiple methods, Isaac demonstrated how he attempted to use multiple methods to deepen students' understanding of proportionality. He frequently made comparisons between the methods and selected a specific task that would help students to see the efficiency and to build meaning for the cross-multiplication method. Isaac's teaching of multiple methods reflected the suggested practices proposed by Durkin et al. (2017), which turned an otherwise meaningless algorithm for students to simply apply into a method that had its particular affordance for solving problems where the factor of change is not immediately obvious, as well as being connected to others.

Conclusion

Isaac's teaching of ratio revealed he had two key considerations for teaching with multiple methods: (i) to demonstrate the efficiency of methods in specific situations, and (ii) to help students make meaning of methods through connections. A cursory analysis of Isaac's teaching with multiple methods may raise several questions about his instructional goals; it might even suggest that he was an ineffective teacher who favoured the use of algorithms over meaningful strategies, such as the unitary method and factors of change. However, upon closer examination

at the item- and set-level, it emerged that Isaac's efforts to emphasise and substantiate the cross-multiplication method were not misguided but instead directed towards preparing students to see its efficiency meaningfully as they approached a problem he had selected. Although the findings in this study are limited to a single teacher and in the context of missing-value problems, we argue that the case of Isaac is valuable because it uncovers that there is more than meets the eye when observing how teachers teach with multiple methods. It demonstrates the underlying complexity of teachers' considerations for teaching with multiple methods, even when the instruction appears to be ill-judged and unproductive at times. Future research should look to document more detailed cases of how teachers teach proportionality with multiple methods. Cases of teachers involving comparison problems and teaching of algorithms would contribute to greater understanding for teacher development programs. Finally, adopting both an item- and set-level analysis could help to reveal these underlying considerations.

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